Numerical Linear Algebra Exam – April, 2017

Do **4** (four) problems

1. (a) Given $A \in \mathbb{C}^{m\times n}$ with $m \geq n$, show that A^*A is nonsingular if and only if A has full rank.

   (b) If $u, v \in \mathbb{C}^m$ and $A = uv^*$, show that $\|A\|_2 = \|u\|_2\|v\|_2$.

2. Assume $S \in \mathbb{C}^{m\times m}$ is skew-Hermitian, so $S^* = -S$.

   (a) Show that the eigenvalues of S are pure imaginary.

   (b) Show that $I - S$ is nonsingular

   (c) Show that the matrix $Q = (I - S)^{-1}(I + S)$ is unitary.

3. If $A \in \mathbb{R}^{m,n}$ with $m \geq n$, $\text{rank}(A) = n$ and $b \in \mathbb{R}^n$.

   (a) Define the least squares solution to $Ax = b$.

   (b) Derive the normal equations for the least squares problem.

   (c) Prove that the unique solution to the least squares problem is $(A^TA)^{-1}A^Tb$.

   (d) Describe how to solve the least squares problem using the SVD decomposition of A.

4. (a) Prove that every square matrix A has a Schur factorization.

   (b) If A is normal (so $A^*A = AA^*$) show that the triangular matrix in its Schur factorization is diagonal.

5. Assume $A \in \mathbb{C}^{m\times m}$

   (a) If A has a collection of m linearly independent eigenvectors, show that A is diagonalizable.

   (b) When A is diagonalizable, prove the Cayley-Hamiliton theorem for A, i.e. A satisfies its own characteristic polynomial.