1st Year Exam: Numerical Analysis, April, 2025.
Do **4** (four) problems.

1. Consider the function $g(x) = e^{-x}$.

(a) Prove that g is a contraction on $G = [\ln 1.1, \ln 3]$.

(b) Prove that g maps $G = [\ln 1.1, \ln 3]$ into $G = [\ln 1.1, \ln 3]$.

(c) Prove that $x_{k+1} = g(x_k)$ converges to an unique fixed point $z \in G = [\ln 1.1, \ln 3]$ for any initial value $x_0 \in G = [\ln 1.1, \ln 3]$.

2. Based on $u_1(x) = 1, u_2(x) = x, u_3(x) = x^2$, use Gram-Schmidt orthogonalization process to compute the three polynomials $w_1(x), w_2(x), w_3(x)$ which are orthonormal on the interval $[0, 1]$ with respect to the inner product $(f, g) = \int_0^1 f(x)g(x)dx$. Using these polynomials, find the best approximation in $P^2[0, 1]$ for $f(x) = x^{\frac{1}{2}}$.

3. Consider the finite difference formula

$$f'(t_j) = \frac{1}{12h}\left[f(t_j - 2h) - 8f(t_j - h) + 8f(t_j + h) - f(t_j + 2h)\right] + O(h^4).$$

(a) Derive this formula by using Taylor's theorem.

(b) Derive this formula by using Lagrange polynomial representation.

4. Assume the numerical quadrature for $\hat{f}(\hat{x})$ on $[0, 1]$ is

$$\hat{J}(\hat{f}) = \int_0^1 \hat{f}(\hat{x})d\hat{x} \approx \hat{Q}(\hat{f}) = \sum_{j=0}^{m} \hat{\alpha}_j \hat{f}(\hat{x}_j).$$

Derive the numerical quadrature of $J(f) = \int_a^b f(x)\,dx$.

5. Consider numerically solving the initial value problem

$$y'(t) = f(t, y),\ 0 < t \le t_f, \qquad \text{with } y(0) = \eta.$$

Assume f is sufficiently differentiable and let h denote the step size. Show that all convergent members of the family of methods

$$y_{n+2} + (\theta - 2)y_{n+1} + (1 - \theta)y_n = \frac{1}{4}h[(6 + \theta)f_{n+2} + 3(\theta - 2)f_n],$$

parameterized by θ, are also A_0-stable.