1st Year Exam: Numerical Analysis, Jan, 2025.
Do **4** (four) problems.

1. $g(x) = \frac{x}{2} + \frac{a}{2x}$ has $x = \sqrt{a}$ as a fixed point where $a > 0$. Suppose we use the fixed point method with a starting point $x_0 > \sqrt{a}$. Use the theory of the fixed point method to determine the condition on x_0 which guarantees

$$|x_{k+1} - \sqrt{a}| \leq \frac{1}{2}|x_k - \sqrt{a}|, \quad k \geq 1.$$

2. Let $f \in C[0,1]$ and let $\sum_{i=1}^{n} w_i f(x_i)\,(n \geq 1)$ be an approximation to $\int_0^1 f(x)dx$. Assume that $0 \leq w_i \leq 1, 0 \leq x_i \leq 1\,(i = 1, 2, \cdots, n)$. Let

$$E_n(f) = \int_0^1 f(x)dx - \sum_{i=1}^{n} w_i f(x_i)$$

be the error in the approximation. Assume that $E_n(p_n) = 0$ for p_n, which denotes any polynomial of degree $\leq n$. Use the Weierstrass approximation theorem to prove that, given $\epsilon > 0$, there is an $N > 0$ such that $|E_n(f)| < \epsilon$ when $n > N$.

3. Given a differentiable function $f(x)$, consider the problem of finding a polynomial $p(x) \in \mathbb{P}^n$ such that

$$p(x_0) = f(x_0), \quad p'(x_i) = f'(x_i), \quad i = 1, 2, \cdots, n,$$

where $x_i, i = 1, 2, \cdots, n$, are distinct nodes. (It is not excluded that $x_1 = x_0$.) Show that the problem has a unique solution and describe how it can be obtained.

4. Let $\{p_i(x)\}_{i=0}^{\infty}$ be the sequence of orthogonal polynomials obtained from the Gram-Schmidt orthogonalization process of $\{x^i\}_{i=0}^{\infty}$. Consider the quadrature formula $Q(f) = \sum_{j=0}^{m} \alpha_j f(x_j)$ for $J(f) = \int_a^b f(x)\rho(x)\,dx$. Prove that $Q(f)$ is exact for polynomials of degree $\leq 2m+1$ if and only if the quadrature nodes $x_j\,(j = 0, 1, \cdots, m)$ are the roots of $p_{m+1}(x)$ and the quadrature weights $\alpha_j = \int_a^b l_j(x)\rho(x)\,dx, (j = 0, 1, \cdots, m)$.

5. Consider numerically solving the initial value problem

$$y'(t) = f(t, y), \; 0 < t \leq t_f, \quad \text{with } y(0) = \eta.$$

Assume f is sufficiently differentiable and let h denote the step size. Show that the method

$$y_{n+2} - y_{n+1} = \frac{1}{4}h(f_{n+2} + 2f_{n+1} + f_n)$$

is A_0 stable.