Numerical Analysis (MAD6407), May, 2026.
Do **4** (four) problems.

- Throughout this exam, $\mathcal{P}_k$ denotes the space of polynomials of degree at most k.
- The recursion for the Chebyshev polynomials of the first kind is given by $T_0(x) = 1$, $T_1(x) = x$, and $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$.

1.

(a) Find the values of α and β that minimize

$$\int_0^1 (x^5 - (1 + \alpha x + \beta x^2))^2 \, \mathrm{d}\, x.$$

(b) Find the error in the best uniform approximation of x^6 over $[-1, 1]$ by $p(x) \in \mathcal{P}_5$. Justify your solution.

2. Prove for any $f \in C[a, b]$ and integer $n \geq 0$, that the best uniform approximation of f in $\mathcal{P}_n$ is unique. You may assume the existence of at least one best uniform approximation of f.

3. Suppose $f \in C^{n+1}[a, b]$ and $p(x) \in \mathcal{P}_n$ interpolates f at $a \leq x_0 \leq x_1 < \ldots < x_n \leq b$. Consider an arbitrary fixed $\hat{x} \in \{x_0, x_1, \ldots, x_n\}$, and derive an exact expression for the error in the first derivative at $\hat{x}$, $f'(\hat{x}) - p'(\hat{x})$.

4. Consider using Newton's method to find $x \in \mathbb{R}^n$ such that $F(x) = 0$, for $F : D \subseteq R^n \to \mathbb{R}^n$, where D is open and convex and F is continuously differentiable on D.

(a) Write down the Newton iteration for finding x_{k+1} from x_k.

(b) Suppose $\bar{x}$ satisfies $F(\bar{x}) = 0$; and $J(x)$, the Jacobian of F evaluated at x satisfies $\|J^{-1}(x)\| \leq \mu$ for some number $\mu > 0$ for all x in a neighborhood $N \subseteq D$ of $\bar{x}$. Show for $x_k \in N$ that

$$\|x_{k+1} - \bar{x}\| \leq \mu \|\bar{x} - x_k\| \cdot \max_{0 \leq t \leq 1} \|J(x_k + t(\bar{x} - x_k)) - J(x_k)\|.$$

5. Consider the finite difference approximation to the first derivative based on $p_1 \in \mathcal{P}_1$ given by $f'(a) \approx p_1'(a) = \frac{f(a+h) - f(a)}{h}$. Define

$$\psi(h) = \frac{f(a + h) - f(a)}{h} = \psi(0) + a_1 h + a_2 h^2 + a_3 h^3 + \ldots.$$

with $\psi(0) = \lim_{h \to 0} \psi(h) = f'(a)$, at least in exact arithmetic. Let $\psi_k(h)$ be the result of k Richardson extrapolations, *e.g.*, $\psi_1(h) = 2\psi(h/2) - \psi(h)$.

(a) Find and prove inductively the general formula for $\psi_k(h)$ based on $\psi_{k-1}(h)$ and $\psi_{k-1}(h/2)$.

(b) State two possible advantages and two possible disadvantages of using Richardson extrapolation.