Numerical Analysis Exam-January 2024
Do **4** (four) problems

1. Derive the three-point formula for the second derivative

$$f''(x) = \frac{1}{h^2}(f(x_0 - h) - 2f(x_0) + f(x_0 + h)) - \frac{h^2}{12} f^{(4)}(\eta),$$

for some $\eta \in [x_0 - h, x_0 + h]$.

2. Consider

$$y' = 4ty; \qquad t \in [0, 1]; \qquad y(0) = 2, \tag{1}$$

which has solution $Y(t) = 2e^{2t^2}$.
(a) Derive an error bound for the forward Euler scheme.
(b) Derive the Taylor method of order 2 for (1).

3. Let $\mathcal{P}_1$ be the space of polynomials of degree at most one. Using the norm $\|u\|_2 = \left(\int_a^b u^2 dx\right)^{\frac{1}{2}}$.
(a) Find the least-squares approximation to $f(x) = x^3$ in $\mathcal{P}_1$ over $[a, b] = [-1, 1]$.
(b) Find the least-squares approximation to $f(x) = x^4$ in $\mathcal{P}_1$ over $[a, b] = [0, 1]$.

4. Consider the fixed point problem $x = f(x)$ where $f(x) = e^{-(3+x)}$.
(a) Assuming all computations are done in exact arithmetic, find the largest open interval in R where the fixed point iteration $x_{k+1} = f(x_k)$ is ensured to converge.
(b) Write a Newton iteration for finding the fixed point.

5. Suppose $f \in C^{n+1}[a, b]$, and let $p \in \mathcal{P}_n$ be a polynomial that interpolates the data $\{(x_i, f(x_i))\}_{i=0}^n$, where $x_0, \cdots, x_n$, are distinct points in $[a, b]$. Consider an arbitrary fixed $x \in [a, b]$, and derive an exact expression for the error $f(x) - p(x)$.