Numerical Analysis Exam-May 2023
Do **4** (four) problems

1. Consider

$$y' = 4ty; \qquad t \in [0,1]; \qquad y(0) = 2, \tag{1}$$

which has solution $Y(t) = 2e^{2t^2}$.

(a) Derive an error bound for the Euler scheme.

(b) Derive the Taylor method of order 2 for (1).

2. Find a, b, c so that the quadrature formula

$$\int_0^2 f(x)dx = af(0) + bf(1) + cf(2)$$

has the largest degree of precision.

3. Suppose $f \in C^{n+1}[a,b]$, and let $p \in \mathcal{P}_n$ be a polynomial that interpolates the data $\{(x_i, f(x_i))\}_{i=0}^n$, where $x_0, \cdots, x_n$, are distinct points in $[a,b]$. Consider an arbitrary fixed $x \in [a,b]$, and derive an exact expression for the error $f(x) - p(x)$.

4. Let $\mathcal{P}_1$ be the space of polynomials of degree at most one. Using the norm $\|u\|_2 = \left(\int_a^b u^2 dx\right)^{\frac{1}{2}}$.

(a) Find the least-squares approximation to $f(x) = x^3$ in $\mathcal{P}_1$ over $[a,b] = [-1,1]$.

(b) Find the least-squares approximation to $f(x) = x^3$ in $\mathcal{P}_1$ over $[a,b] = [0,1]$.

5. Let $G = [0,2]$ and

$$g(x) = \frac{1}{3}(\frac{x^3}{3} - x^2 - \frac{5x}{4} + 4).$$

Use the contraction mapping theorem to prove that if $x_0 \in G$, then the sequence defined by $x_{k+1} = g(x_k) (k = 0, 1, \cdots)$ converges to a unique fixed point $z \in G$.

(b) Consider the fixed point iteration method $x_{k+1} = g(x_k) (k = 0, 1, \cdots)$ for solving the nonlinear equation $f(x) = 0$. Consider choosing an iteration function of the form

$$g(x) = x - af(x) - b(f(x))^2 - c(f(x))^3,$$

where a, b, c are parameters to be determined. Assume f is sufficiently differentiable and the corresponding iterations $x_{k+1} = g(x_k)$ converge to a unique fixed point z. Find expressions for the parameters a, b, c in terms of functions of z such that the iteration method is of fourth order.