1st Year Exam: Numerical Analysis, May, 2022.
Do **4** (four) problems.

1. **(a)** Let $G = [0, 2]$ and

$$g(x) = \frac{1}{3}(\frac{x^3}{3} - x^2 - \frac{5}{4}x + 4).$$

Use the contraction mapping theorem to prove that if $x_0 \in G$, then the sequence defined by $x_{k+1} = g(x_k), (k = 0, 1, \cdots)$ converges to a unique fixed point $z \in G$.

(b) Consider the fixed point iteration method $x_{k+1} = g(x_k)$, $k = 0, 1, \cdots$ for solving the nonlinear equation $f(x) = 0$. Consider choosing an iteration function of the form

$$g(x) = x - af(x) - b(f(x))^2 - c(f(x))^3,$$

where a, b, and c are parameters to be determined. Assume f is sufficiently differentiable and the corresponding iterations $x_{k+1} = g(x_k)$ converge to a unique fixed point z. Find expressions for the parameters a, b, and c in terms of functions of z such that the iteration method is of fourth order.

2. Consider $f(t) = \sin(t)$.

(a) Without using orthogonal polynomials, find the best approximation $p_1(t) \in P^2[-1, 1]$ to $f(t) \in C[-1, 1]$ with respect to the L^2 norm.

(b) Find the Taylor polynomial approximation $p_2(t)$ of degree 3 at $t = 0$.

(c) Find the Lagrange polynomial approximation $p_3(t)$ of degree 3 that interpolates $f(t)$ at $t = -1, -\frac{1}{3}, \frac{1}{3}, 1$.

3. Given a differentiable function $f(x)$, consider the problem of finding a polynomial $p(x) \in \mathbb{P}^n$ such that

$$p(x_0) = f(x_0), \quad p'(x_i) = f'(x_i), \quad i = 1, 2, \cdots, n,$$

where $x_i, i = 1, 2, \cdots, n$, are distinct nodes. (It is not excluded that $x_1 = x_0$.) Show that the problem has a unique solution and describe how it can be obtained.

4. Let $Q_m(f)$ denote the m-point Gaussian quadrature rule over the interval $[a, b]$ and with continuous weight function $\rho(x) \geq 0$, that is,

$$Q_m(f) = \sum_{i=1}^{m} \alpha_i f(x_i) \approx J(f) = \int_a^b \rho(x) f(x)\, dx.$$

Show that, if a and b are finite and f is continuous, then $Q_m(f) \to J(f)$ as $m \to \infty$.

5. Consider numerically solving the initial value problem

$$y'(t) = f(t, y), \ 0 < t \leq t_f, \quad \text{with } y(0) = \eta.$$

Assume f is sufficiently differentiable and let h denote the step size. Show that all convergent members of the family of methods

$$y_{n+2} + (\theta - 2)y_{n+1} + (1 - \theta)y_n = \frac{1}{4}h[(6 + \theta)f_{n+2} + 3(\theta - 2)f_n],$$

parameterized by θ, are also A_0-stable.