# First-year Analysis Examination
# Part Two
# May 2025

---

Answer FOUR questions in detail.
State carefully any results used without proof.

---

1. The sequence $(f_n)_{n=0}^{\infty}$ of real-valued functions on $[0,1]$ is uniformly bounded and converges to 0 pointwise. If each $f_n$ is (i) continuous, (ii) Riemann integrable, (iii) Lebesgue integrable, does it follow that $\int_0^1 f_n(t)\mathrm{d}t \to 0$? In each of the three cases, a proof or counterexample should be given.

2. Let $(f_n)_{n=0}^{\infty}$ be a sequence of continuous non-negative functions on $[0,1]$ converging pointwise to 0. Prove that if the *extra condition*

$$\forall t \in [0,1] \;\; f_0(t) \geqslant f_1(t) \geqslant ... \geqslant f_n(t) \geqslant ...$$

is satisfied then $f_n \to 0$ uniformly on $[0,1]$. Show that this conclusion can fail if the *extra condition* is removed.

3. Let the continuous real-valued function $f$ on $[0,1]$ satisfy

$$\forall n \in \mathbb{N} \;\; \int_0^1 f(t)(1-t)^n \mathrm{d}t = 0.$$

Deduce as much as possible about $f$.

4. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Let $(A_n)$ be a sequence in $\mathcal{A}$ and define

$$\liminf A_n = \bigcup_N \bigcap_{n \geqslant N} A_n.$$

Prove that

$$\mu(\liminf A_n) \leqslant \liminf \mu(A_n).$$

5. Let $f(x) = \sin x / x$ for $x > 0$. Show that if $n$ is a positive integer then

$$\int_{2n\pi}^{(2n+1)\pi} \frac{\sin x}{x} \mathrm{d}x \geqslant \frac{2}{(2n+1)\pi}.$$

Hence explain why $f$ is *not* Lebesgue integrable on $(a, \infty)$ for any $a > 0$.

---