First-year Analysis Examination
Part Two
January 2026

Answer FOUR questions in detail.
State carefully any results used without proof.

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1. Prove that the bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable in each of the following cases: (a) f is increasing; (b) f is continuous.

2. Let $(f_n)_{n=0}^{\infty}$ be a uniformly convergent sequence of continuous real-valued functions on the metric space X and let $(x_n)_{n=0}^{\infty}$ be a convergent sequence in X. Must the real sequence $(f_n(x_n))_{n=0}^{\infty}$ converge? Proof or counterexample.

3. Let $\mathcal{F}$ be an equicontinuous family of real-valued functions on the connected metric space X and when $x \in X$ write $\mathcal{F}_x = \{f(x) : f \in \mathcal{F}\} \subseteq \mathbb{R}$. Prove: if $\mathcal{F}_x$ is bounded for *some* $x \in X$ then $\mathcal{F}_x$ is bounded for *every* $x \in X$.

4. Let $X \subseteq \mathbb{R}$ have finite Lebesgue measure $\lambda(X)$. Prove:
(a) for each $\varepsilon > 0$ some open set $U \subseteq \mathbb{R}$ satisfies $U \supseteq X$ and $\lambda(U) < \lambda(X)+\varepsilon$;
(b) some Borel set $B \subseteq \mathbb{R}$ satisfies $U \supseteq X$ and $\lambda(B) = \lambda(X)$.

5. Let the sequence $(f_n)_{n=0}^{\infty}$ of non-negative integrable functions converge pointwise to f on some measure space $(\Omega, \mathcal{A}, \mu)$. If it is assumed that that the integrals $\int_\Omega f_n \mathrm{d}\mu$ are all at most $M < \infty$, does it follow that f is integrable and that $\int_\Omega f \mathrm{d}\mu \leqslant M$? Proof or counterexample.
