First-year Analysis Examination
Part Two
August 2024

Answer FOUR questions in detail.
State carefully any results used without proof.

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1. (i) The function $f : [-1, 1] \to \mathbb{R}$ is *Riemann* integrable and $F : [-1, 1] \to \mathbb{R}$ is defined by the rule

$$F(t) = \int_0^t f(u)\mathrm{d}u.$$

Prove that F is continuous at 0.

(ii) Does the same conclusion follow if f is instead assumed to be *Lebesgue* integrable? Explain.

2. Let $A = \{a_n : n \in \mathbb{N}\}$ be a subset of the metric space X. For each $n \in \mathbb{N}$ let $f_n : X \to \mathbb{R}$ be continuous except perhaps at a_n.
(i) Prove that if $f_n \to f$ *uniformly* on X then the set of discontinuities of f is finite or countably infinite.
(ii) Show by example that if $f_n \to f$ *pointwise* on X then f can have uncountably many discontinuities.

3. Let $f : [0, 1] \to \mathbb{R}$ be continuous; assume that

$$\int_0^1 f(t)t^n \mathrm{d}t = 0$$

for each integer $n > 1$. Prove that the function f is constantly zero.

4. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space that is *complete* in the sense that each subset of every μ-null set lies in $\mathcal{A}$. Let $f : \Omega \to \mathbb{R}$ be measurable and let $A \subseteq \Omega$ be μ-null. Prove that if $g : \Omega \to \mathbb{R}$ is defined by changing the values of f at points of A, then g is measurable.

5. On the measure space Ω, let $(f_n : n \in \mathbb{N})$ be a sequence of μ-integrable functions that converges uniformly to the function f. Prove that if $\mu(\Omega)$ is finite then f is μ-integrable, and show by example that f can fail to be μ-integrable if $\mu(\Omega)$ is infinite.
