First-year Analysis Examination
Part Two
May 2024

Answer FOUR questions in detail.
State carefully any results used without proof.

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1. Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable, let g be a real-valued function on $[a, b]$, and let $D = \{t \in [a, b] : f(t) \neq g(t)\}$.
(i) Show by example that if D is countably infinite then g can fail to be Riemann integrable.
(ii) Prove that if D is finite then g must be Riemann ntegrable.

2. Prove Dini's theorem that, if the sequence $(f_n)_{n=0}^{\infty}$ of continuous real-valued functions on a compact space decreases pointwise to zero, then the convergence is uniform. Show by example that uniform convergence can fail if *decreases pointwise* is replaced by *converges pointwise*.

3. State the Weierstrass approximation theorem. Determine exactly which continuous functions $f : [-1, 1] \to \mathbb{R}$ satisfy the requirement that for each strictly positive integer n,

$$\int_{-1}^{1} f(t)t^n \mathrm{d}t = 0.$$

4. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $(A_n)_{n=1}^{\infty}$ be a sequence in $\mathcal{A}$. For each positive integer N, let B_N be the set containing exactly those $\omega \in \Omega$ that lie in A_n for at least N values of n. By considering the series $\sum_{n=1}^{\infty} 1_{A_n}$ of indicator functions, prove that

$$N\,\mu(B_N) \leqslant \sum_{n=1}^{\infty} \mu(A_n).$$

5. Let the non-negative function $f : \mathbb{R} \to \mathbb{R}$ be integrable with respect to Lebesgue measure λ and for each real t define

$$F(t) = \int_0^t f \mathrm{d}\lambda.$$

Prove that the function F is continuous. Must F be uniformly continuous? Explain.
