First-year Analysis Examination
Part Two
January 2024

Answer FOUR questions in detail.
State carefully any results used without proof.

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1. Prove that the function $f : [a, b] \to \mathbb{R}$ is Riemann integrable iff for each $\varepsilon > 0$ there exist Riemann integrable functions $\ell, u : [a, b] \to \mathbb{R}$ such that $\ell \leqslant f \leqslant u$ pointwise on $[a, b]$ and $\int_a^b (u - \ell) < \varepsilon$.

2. For each positive integer n let $f_n ; [0, 1] \to [0, 1]$ be a Riemann integrable function. Which (if any) of the following conditions is sufficient to ensure that $\int_0^1 f_n(t)\mathrm{d}t \to 0$ as $n \to \infty$?
(i) $f_n \to 0$ uniformly on $[0, 1]$; (ii) $f_n \to 0$ pointwise on $[0, 1]$.

3. To each continuous function $f : [0, 1] \to \mathbb{R}$ associate the sequence $(a(f)_n)_{n=0}^{\infty}$ defined by

$$a(f)_n = \int_0^1 f(t)t^n \mathrm{d}t.$$

Prove that the map $f \mapsto a(f)$ is injective.

4. Let the bounded set $X \subseteq \mathbb{R}$ be Lebesgue measurable. Prove that:
(i) for each $\varepsilon > 0$ there exists an open set $U \supseteq X$ whose Lebesgue measure $\lambda(U)$ is less than $\lambda(X) + \varepsilon$;
(ii) there exists a Borel set $B \supseteq X$ such that $\lambda(B) = \lambda(X)$.

5. Let $(f_n)_{n=0}^{\infty}$ be a sequence of functions on the measurable space $(X, \mathcal{A})$ with values in $[0, \infty)$. Prove that each of the following subsets of X is measurable:
(i) $B = \{x : \text{the sequence } (f_n(x))_{n=0}^{\infty} \text{ is bounded}\}$;
(ii) $C = \{x : \text{the series } \sum_{n=0}^{\infty} f_n(x) \text{ is convergent}\}$.
