First-year Analysis Examination
Part Two
August 2023

Answer FOUR questions in detail.
State carefully any results used without proof.

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1. Let (f_n) be a sequence of continuous functions from $[0, 1]$ to $[0, 1]$. For each of the following, give a proof (if true) or a counterexample (if false):
(i) if $f_n(t)$ decreases pointwise to 0 then $\int_0^1 f_n(t)\mathrm{d}t \to 0$;
(ii) if $f_n(t)$ only converges pointwise to 0 then $\int_0^1 f_n(t)\mathrm{d}t \to 0$.

2. For $0 < t < 1$ and for n a positive integer, let $f_n(t) = t^n$. (i) Show that the sequence (f_n) converges uniformly on each compact subset of $(0, 1)$.
(ii) Show that (f_n) does not converge uniformly on $(0, 1)$.

3. Let the function $f : [1, \infty) \to \mathbb{R}$ be continuous, with $\lim_{t\to\infty} f(t) = A \in \mathbb{R}$. Prove there exists a sequence (p_n) of polynomials such that $p_n(1/t) \to f(t)$ uniformly for $t \geqslant 1$.

4. Let $A \subseteq [0, 1]$ have positive Lebesgue measure: that is, $\lambda(A) > 0$. Prove that if $0 < \alpha < \lambda(A)$ then there exists a Lebesgue measurable set $A_\alpha \subseteq A$ such that $\lambda(A_\alpha) = \alpha$.

5. State the Monotone Convergence Theorem and 'Fatou's Lemma'. Deduce *one* of these from the other.
