First-year Analysis Examination
Part Two
May 2023

Answer FOUR questions in detail.
State carefully any results used without proof.

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1. Prove that the function $f : [a, b] \to \mathbb{R}$ is Riemann-integrable if and only if for each $\epsilon > 0$ there exist Riemann-integrable functions ℓ and u on $[a, b]$ such that $\ell \leq f \leq u$ and $\int_a^b (u - \ell) < \epsilon$.

2. Let $f : [0, 1] \to \mathbb{R}$ be continuous; for each $n \in \mathbb{N}$ and each $t \in [0, 1]$ define $f_n(t) = f(t)t^n$. Prove that the sequence $(f_n : n \in \mathbb{N})$ converges uniformly on $[0, 1]$ if and only if $f(1)$ has a specific value, which should be found.

3. Let $f : [-1, 1] \to \mathbb{R}$ be continuous. Prove that if

$$\int_{-1}^{1} f(t)t^n \mathrm{d}t = 0$$

whenever the natural number n is odd, then the function f is even.

4. Let $(f_n : n \in \mathbb{N})$ be a sequence of measurable real-valued functions on some measurable space. Prove that each of these three sets is measurable:

$C = \{\omega :$ the sequence $f_n(\omega)$ is Cauchy$\}$;
$D = \{\omega :$ the sequence $f_n(\omega)$ has all terms different$\}$;
$E = \{\omega :$ the sequence $f_n(\omega)$ is eventually constant$\}$.

5. The bounded function $f : [0, 1] \times [0, 1] \to \mathbb{R}$ is separately continuous: that is, $f(x, y)$ is continuous in x when y is fixed and continuous in y when x is fixed. Prove the continuity of the function $F : [0, 1] \times [0, 1] \to \mathbb{R}$ defined by

$$F(x) = \int_0^1 f(x, y)\mathrm{d}y.$$
