First-year Analysis Examination
Part Two
January 2023

Answer FOUR questions in detail.
State carefully any results used without proof.

---

1. Let f be a real-valued function on $[0, 1]$ and consider the statements:
(i) for each positive integer n the function f is Riemann-integrable on $[1/n, 1]$ and the sequence of integrals $\int_{1/n}^{1} f$ converges to a real number as $n \to \infty$;
(ii) the function f is Riemann-integrable on $[0, 1]$.
Prove that (ii) implies (i); and show by example that (i) $\Rightarrow$ (ii) can fail.

2. Let X be a metric space on which (f_n) and (g_n) are uniformly convergent sequences of continuous real-valued functions.
(i) Prove that if X is compact then $(f_n g_n)$ is uniformly convergent.
(ii) Show that the conclusion in (i) can fail when X is not compact.

3. Let $f : [0, 1] \times [0, 1] \to \mathbb{R}$ be continuous. By first considering polynomials in two variables, prove that

$$\int_0^1 \Big(\int_0^1 f(x, y)\, \mathrm{d}x \Big) \mathrm{d}y = \int_0^1 \Big(\int_0^1 f(x, y)\, \mathrm{d}y \Big) \mathrm{d}x.$$

4. Prove that if (f_n) is a sequence of measurable functions such that $f_n \to f$ on some measure space, then f is measurable. Hence, or otherwise, show that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable then its derivative f' is Borel measurable.

5. In each of the following cases, give (if possible and with justification) a sequence (f_n) of Lebesgue integrable functions from $\mathbb{R}$ to $\mathbb{R}$ such that:
(i) $f_n \to 0$ pointwise but $\int f_n \mathrm{d}\lambda \to 1$ as $n \to \infty$;
(ii) $f_n \to 0$ pointwise but $\int f_n \mathrm{d}\lambda \to \infty$ as $n \to \infty$
where λ denotes Lebesgue measure.
