First-year Analysis Examination
Part Two
August 2022

Answer FOUR questions in detail.
State carefully any results used without proof.

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1. Let $f; [0, 1] \to \mathbb{R}$ satisfy the following rules: if t is irrational then $f(t) = 0$; if $t = m/n$ is a rational in lowest terms then $f(t) = 1/n$. Prove that f is Riemann integrable and calculate its integral.

2. Let $\mathcal{F} \subseteq C(X)$ be equicontinuous and let A be the set comprising all points of X at which $\mathcal{F}$ is bounded; that is, $a \in A$ exactly when there exists K such that $|f(a)| < K$ whenever $f \in \mathcal{F}$. Prove that $A \subseteq X$ is both closed and open.

3. Let the function $f : \mathbb{R} \to \mathbb{R}$ be differentiable to all orders and denote its nth derivative by f_n. Assume that the sequence $(f_n)_{n=0}^{\infty}$ converges uniformly to the function g on $\mathbb{R}$ and deduce as much as possible about g.

4. Prove that if the real-valued functions f and g on the same space are measurable, then so are their pointwise sum $f + g$ and pointwise product fg.

5. For each $n \in \mathbb{N}$ let $f_n : [0, 1] \to [0, 1]$ be continuous and assume that $f_n \to 0$ pointwise as $n \to \infty$. Does it follow that

$$\int_0^1 f_n(t)\mathrm{d}t \to 0 \quad \text{as} \quad n \to \infty?$$

Does your answer change if *continuous* is replaced by *Riemann integrable*?
