First-year Analysis Examination
Part Two
May 2022

Answer FOUR questions in detail.
State carefully any results used without proof.

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1. Let $f : [a, b] \to \mathbb{R}$ have the property that for each $\varepsilon > 0$ there exist Riemann integrable functions ℓ and u on $[a, b]$ such that $\ell \leqslant f \leqslant u$ and $\int_a^b (u - \ell) < \varepsilon$. Does it follow that f is Riemann integrable on $[a, b]$? Proof or counterexample.

2. Suppose $\big(p_n\big)_{n=0}^{\infty}$ is a sequence of polynomials in one variable.

(i) Assume that $p_n \to f$ uniformly on $[0, 1]$ as $n \to \infty$; deduce as much as possible about the function $f : [0, 1] \to \mathbb{R}$.

(ii) Assume that $p_n \to f$ uniformly on $\mathbb{R}$ as $n \to \infty$; deduce as much as possible about the function $f : \mathbb{R} \to \mathbb{R}$.

3. Let the continuous function $f : [0, 1] \to \mathbb{R}$ satisfy

$$\int_0^1 f(t) t^n \mathrm{d}t = 1/(n + 2)$$

for all but finitely many positive integers n. Deduce as much as possible about f.

4. Let $\big(f_n\big)_{n=0}^{\infty}$ be a sequence of measurable real-valued functions on some measurable space. Prove that each of the following sets is measurable:

(i) $A = \{\omega : \sum_{n=0}^{\infty} f_n(\omega) \text{ is absolutely convergent}\}$;

(ii) $C = \{\omega : \sum_{n=0}^{\infty} f_n(\omega) \text{ is conditionally convergent}\}$.

5. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space; let $A_0 \subseteq A_1 \subseteq \cdots$ be an increasing sequence in $\mathcal{A}$ such that $\cup_{n=0}^{\infty} A_n = \Omega$; let $f : \Omega \to \mathbb{R}$ be measurable; and let L be a real number. Consider the statements:

(i) f is integrable on Ω and $\int_\Omega f \mathrm{d}\mu = L$;

(ii) f is integrable on each A_n and $\int_{A_n} f \mathrm{d}\mu \to L$ as $n \to \infty$.

Prove that (i) implies (ii) and decide (with proof or counterexample) whether (ii) implies (i).
