First-year Analysis Examination
Part Two
January 2022

Answer FOUR questions in detail.
State carefully any results used without proof.

---

1. For each integer $n \geqslant 1$ let $\chi_n = n1_{[0,1/n]}$ be n times the indicator function of the interval $[0, 1/n]$. Prove that if $f : [0,1] \to \mathbb{R}$ is continuous then

$$\lim_{n\to\infty} \int_0^1 \chi_n(t) f(t)\mathrm{d}t = f(0).$$

2. Let $a > 0$. When n is a positive integer and $t \geqslant 0$ write

$$f_n(t) = \frac{\sin nt}{1 + nt}.$$

Is the sequence $(f_n : n > 0)$ uniformly convergent on $[a, \infty)$? On $[0, \infty)$?

3. Let $f : [0,1] \to \mathbb{R}$ be continuous. Prove that if

$$\int_0^1 f(t) t^{n+\frac{1}{2}} \mathrm{d}t = 0$$

for each positive integer n then f vanishes identically.

4. When $(f_n : n > 0)$ is a sequence of measurable real-valued functions on some measurable space, show that the following sets are measurable:
(i) $\{\omega : f_n(\omega)$ alternates in sign$\}$
(ii) $\{\omega : f_n(\omega)$ is eventually rational$\}$
(iii) $\{\omega : \sum_{n>0} f_n(\omega)$ is absolutely convergent$\}$.

5. State the Monotone Convergence Theorem and the Fatou Lemma, and deduce one of these from the other.
