First-year Analysis Examination
Part Two
August 2021

Answer FOUR questions in detail.
State carefully any results used without proof.

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1. Let f_A be the indicator function of the countably infinite set $A \subseteq [0,1]$: that is, $f_A(t)$ is 1 when $t \in A$ and 0 when $t \in [0,1] \setminus A$.
(i) Exhibit (with brief justification) such an A for which f_A *is* Riemann integrable, or prove that no such A exists.
(ii) Exhibit (with brief justification) such an A for which f_A *is not* Riemann integrable, or prove that no such A exists.

2. For each $n > 0$ let the function $f_n : X \to \mathbb{R}$ be continuous at all but finitely many points. Prove that if $f_n \to f$ uniformly on X, then f is continuous at all but countably many points.

3. $(f_n : n > 0)$ is an equicontinuous sequence of real-valued functions on a compact space. Prove that if $f_n \to f$ pointwise then $f_n \to f$ uniformly.

4. Let $(f_n : n > 0)$ be a sequence of measurable functions on Ω. Prove that each of the following sets is measurable:
(i) $\{\omega \in \Omega : \text{the sequence } f_n(\omega) \text{ is eventually constant}\}$;
(ii) $\{\omega \in \Omega : \text{the values } f_n(\omega) \text{ are all different}\}$.

5. Let $(f_n : n > 0)$ be a sequence of non-negative integrable functions that converges pointwise to f. Prove that if

$$\int_\Omega f_n \, \mathrm{d}\mu \to \int_\Omega f \, \mathrm{d}\mu$$

then

$$\int_\Omega |f - f_n| \, \mathrm{d}\mu \to 0.$$

[It might help to consider the positive part $(f - f_n)^+ = \max\{0, f - f_n\}$.]
