First-year Analysis Examination
Part Two
May 2021

Answer FOUR questions in detail.
State carefully any results used without proof.

---

1. For each $n \in \mathbb{N}$ let $f_n : [0,1] \to \mathbb{R}$ be continuous. Assume that for each $t \in [0,1]$ the sequence $(f_n(t) : n \in \mathbb{N})$ converges to 0 monotonically. Does it follow that $\int_0^1 f_n(t)\mathrm{d}t \to 0$? Does the answer change if the word 'monotonically' is removed?

2. Let $f : [-1,1] \to \mathbb{R}$ be continuous and assume that $\int_{-1}^1 f(t)t^n \mathrm{d}t = 0$ for each even integer $n \geqslant 0$. Deduce as much as is possible about the nature of the function f.

3. Let $(f_n : n \in \mathbb{N})$ be a sequence of measurable real-valued functions on the same measurable space. In each of the following cases, show that the set of points ω satisfying the stated condition is a measurable set:
(i) the sequence $(f_n(\omega) : n \in \mathbb{N})$ is unbounded;
(ii) $(f_n(\omega) : n \in \mathbb{N})$ is not strictly monotonic;
(iii) $(f_n(\omega) : n \in \mathbb{N})$ revisits its initial value $f_0(\omega)$ infinitely often.

4. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space on which $f \geqslant 0$ is an integrable function. Prove that for each $\varepsilon > 0$ there exists $\delta > 0$ such that each $A \in \mathcal{A}$ with $\mu(A) < \delta$ satisfies $\int_A f \mathrm{d}\mu < \varepsilon$.

5. Consider these two statements about the function $f : \mathbb{R} \to \mathbb{R}$:
(a) f is integrable with respect to Lebesgue measure λ on $\mathbb{R}$;
(b) f is Lebesgue integrable on $[-n,n]$ for each integer $n > 0$ and the sequence of integrals $\int_{-n}^n f \mathrm{d}\lambda$ converges.
Prove that (a) $\Rightarrow$ (b) and show by example that (b) $\Rightarrow$ (a) can fail.
