First-year Analysis Examination
Part One
January 2025

Answer FOUR questions in detail.
State carefully any results used without proof.

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1. Let (a_n) be a bounded sequence of strictly positive real numbers. Prove that

$$\limsup(a_n^{1/n}) \leqslant \limsup(a_{n+1}/a_n).$$

Hence prove that if $a_{n+1}/a_n \to \ell \in \mathbb{R}$ then $a_n^{1/n} \to \ell$ (as $n \to \infty$).

2. The diameter D of the bounded metric space X is defined by

$$D = \sup\{d(a, b) : a, b \in X\}.$$

Prove that if X is compact then $D = d(a_0, b_0)$ for some $a_0, b_0 \in X$.

3. The compact metric space X contains $A_1 \supseteq A_2 \supseteq \dots$ as a decreasing sequence of closed sets.
(i) Prove that if each A_n is nonempty then so is the intersection $A = \cap_{n=1}^{\infty} A_n$.
(ii) Prove that if $U \subseteq X$ is open and contains A then U contains some A_n.

4. Let $f : X \to Y$ be a map between metric spaces and consider the statements:
(a) if (x_n) is any Cauchy sequence in X then $(f(x_n))$ is Cauchy in Y;
(b) f is uniformly continuous.
Does (a) imply (b)? Does (b) imply (a)? In each case, give a proof if so, a counterexample if no.

5. Let $\ell \in \mathbb{R}$, let $f : \mathbb{R} \to \mathbb{R}$, and assume that $f'(t) \to 0$ as $\mathbb{R} \ni t \to \infty$. Prove that if $f(n) \to \ell$ as $\mathbb{N} \ni n \to \infty$ then $f(t) \to \ell$ as $\mathbb{R} \ni t \to \infty$.
