First-year Analysis Examination
Part One
August 2024

Answer FOUR questions in detail.
State carefully any results used without proof.

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1. Let $f : [0,1] \to [0,1]$ be (weakly) increasing. Explain why the set

$$A = \{t \in [0,1] : t \leqslant f(t)\}$$

has a supremum. Prove that the function f has a fixed point.

2. Let A be a dense subset of the metric space M. Prove that M is complete if every Cauchy sequence in A converges in M.

3. Let $f : X \to Y$ be an isometry (a distance-preserving map) between metric spaces. In each of the following cases, decide whether the image $f(X)$ must be closed in Y: (i) X is compact; (ii) X is complete.

4. Does there exist an injective continuous real-valued function on the open unit square $U = (0,1) \times (0,1)$? If so, give an example; if no, prove that none exists.

5. The function $f : (0,1) \to \mathbb{R}$ is differentiable, with bounded derivative. Must $\lim_{t \to 0+} f(t)$ exist in $\mathbb{R}$? Proof or counterexample.
