First-year Analysis Examination
Part One
May 2024

Answer FOUR questions in detail.
State carefully any results used without proof.

---

1. Assume without proof that if (a_n) and (b_n) are bounded real sequences then

$$\limsup(a_n + b_n) \leqslant \limsup a_n + \limsup b_n.$$

(i) Show by example that this inequality can be strict.
(ii) Prove that equality holds if (a_n) is convergent.

2. Let A be a subset of the metric space M. Assume that A satisfies the following condition:

$$\forall X \subseteq M \ (A \cap X = \emptyset \Rightarrow A \cap \overline{X} = \emptyset).$$

Does it follow that A is open? Proof or counterexample.

3. Let A and B be nonempty compact subsets of the metric space M. Prove that there exist $a \in A$ and $b \in B$ such that

$$d(a, b) = \inf\{d(x, y) : x \in A, y \in B\}.$$

4. Does there exist a one-to-one continuous function from the unit square $\square = [0, 1] \times [0, 1]$ to $\mathbb{R}$? If so, exhibit one; if no, prove not.

5. Let $f : (0, \infty) \to \mathbb{R}$ be differentiable and let $b \in \mathbb{R}$. Prove that if $f'(t) \to b$ as $t \to \infty$ then $f(t)/t \to b$ as $t \to \infty$. [It might help to choose and fix $a > 0$.]
