First-year Analysis Examination
Part One
January 2024

Answer FOUR questions in detail.
State carefully any results used without proof.

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1. Let $f : (0, \infty) \to \mathbb{R}$ be continuous. Prove that if

$$\limsup_{n\to\infty} f(n) > 0 > \liminf_{n\to\infty} f(n)$$

then f has infinitely many zeros. Does the same conclusion hold if one of the strict inequalities is replaced by a weak inequality?

2. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and assume that $f^{-1}(y) \subseteq \mathbb{Q}$ whenever $y \in \mathbb{R} \setminus \mathbb{Q}$. What can be deduced about f?

3. Let the function $f : (0, 1) \to \mathbb{R}$ be *uniformly* continuous.
(i) Prove that f extends to a continuous function $F : [0, 1] \to \mathbb{R}$.
(ii) Does the same conclusion follow if f is merely continuous? What if f is continuous and bounded? Justify your assertions.

4. Let $f : X \to Y$ be a map between metric spaces and consider the following two statements:
(i) if (x_n) is a Cauchy sequence in X, then $(f(x_n))$ is Cauchy in Y;
(ii) f is uniformly continuous.
For each implication (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i), give a proof (if true) or a counterexample (if false).

5. The function $f : \mathbb{R} \to \mathbb{R}$ is differentiable, except perhaps at $a \in \mathbb{R}$. Prove that if the real limit $\lim_{t\to a} f'(t)$ exists then f is in fact differentiable at a.
