First-year Analysis Examination
Part One
August 2023

Answer FOUR questions in detail.
State carefully any results used without proof.

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1. Let the nonempty subsets $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$ be bounded above; define $A + B = \{a + b : a \in A, b \in B\}$ and $AB = \{ab : a \in A, b \in B\}$.
(i) Prove that $\sup(A + B) = \sup A + \sup B$.
(ii) Show by example that '$\sup(AB) = \sup A \sup B$' can fail to hold.

2. Let U be an open subset of a metric space and let A be an arbitrary subset of the same space. Prove that when closures are taken in this same space,

$$U \cap \overline{A} \subseteq \overline{U \cap A}.$$

3. Prove that the square-root function $f : [0, \infty) \to [0, \infty) : t \mapsto \sqrt{t}$ is uniformly continuous.

4. Let (X, d) be a compact metric space and $f : X \to X$ a continuous function. Prove that if f has no fixed point (that is, there exists no $p \in X$ such that $f(p) = p$) then $d(x, f(x)) \geq m$ for some $m > 0$ and all $x \in X$.

5. Let the real-valued function f be continuous on $[0, \infty)$ and differentiable on $(0, \infty)$; assume that $f(0) = 0$ and that the derivative f' is increasing. Prove that $F : (0, \infty) \to \mathbb{R}$ defined by $F(x) = f(x)/x$ is increasing in the same sense (weakly or strictly).
