First-year Analysis Examination
Part One
May 2023

Answer FOUR questions in detail.
State carefully any results used without proof.

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1. Let the continuous function $f : (0, \infty) \to \mathbb{R}$ satisfy

$$\liminf f(n) < 0 < \limsup f(n).$$

Prove that f has infinitely many zeros.

2. Fix any metric space $\mathbb{T}$ with exactly two points. Prove that the metric space X is connected if and only if each continuous function f from X to $\mathbb{T}$ is constant.

3. Let $B(X)$ be the set of all bounded real-valued functions on the set X. Prove that the formula

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in X\}$$

defines a metric d on $B(X)$ that makes $B(X)$ into a *complete* metric space.

4. Let $A_1 \supseteq A_2 \supseteq \dots$ be a decreasing sequence of nonempty subsets of $\mathbb{R}^N$ (with its standard Euclidean metric).
(i) Prove that if each A_n is *compact* then $\cap_{n=1}^{\infty} A_n$ is nonempty.
(ii) Show by example that $\cap_{n=1}^{\infty} A_n$ can be empty if each A_n is only *closed*.

5. Let $f : (a, b) \to \mathbb{R}$ be differentiable, with bounded derivative. Prove the existence of the real limits

$$\lim_{x \to a+} f(x) \quad \text{and} \quad \lim_{x \to b-} f(x).$$
