First-year Analysis Examination
Part One
January 2023

Answer FOUR questions in detail.
State carefully any results used without proof.

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1. Let (a_n) and (b_n) be bounded real sequences. Prove that

$$\limsup(a_n + b_n) \leqslant \limsup a_n + \limsup b_n$$

with equality when (a_n) converges.

2. Let A be a subset of the metric space (M, d) and define $d_A : M \to \mathbb{R}$ by $d_A(x) = \inf\{d(a, x) : a \in A\}$. Prove that:
(i) d_A is uniformly continuous;
(ii) $\{x \in M : d_A(x) = 0\} = \overline{A}$.

3. Let X and Y be metric spaces; let $f : X \to Y$ be a continuous bijection, with inverse $g : Y \to X$. For each of the following statements, provide a proof if true or a counterexample (with justification) if false:
(i) if X is compact then g is continuous;
(ii) if Y is compact then g is continuous.

4. Let X and Y be metric spaces; let $f : X \to Y$ be a continuous bijection, with uniformly continuous inverse $g : Y \to X$. For each of the following statements, provide a proof if true or a counterexample if false:
(i) if X is complete then Y is complete;
(ii) if Y is complete then X is complete.
[Be sure to demonstrate uniform continuity of g in any counterexample.]

5. The function $f : \mathbb{R} \to \mathbb{R}$ is differentiable, with $|f'| \leqslant 1$ everywhere. Prove that if $\varepsilon > 0$ is sufficiently small (how small?) then the function $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = x + \varepsilon f(x)$ is injective.
