First-year Analysis Examination
Part One
August 2022

Answer FOUR questions in detail.
State carefully any results used without proof.

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1. Let $(s_n : n > 0)$ be a bounded real sequence and write $\sigma_n = (s_1 + \cdots + s_n)/n$ for each $n > 0$. Prove that

$$\limsup_n \sigma_n \leqslant \limsup_n s_n.$$

Hence, or otherwise, show that if $s_n \to s$ then $\sigma_n \to s$ also.

2. Let X be a metric space; let $A \subseteq X$ be dense and let $U \subseteq X$ be open. Prove that $U \subseteq \overline{A \cap U}$, where the overline denotes closure.

3. Let A be a subset of the metric space X and let $f : X \to \mathbb{R}$ be continuous.
(i) Prove that if A is compact, then f is bounded on A.
(ii) Show by example that if instead A is closed and bounded, then f can fail to be bounded on A.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous; assume that $\liminf_{n\to\infty} f(n) = -1$ and that $\limsup_{n\to\infty} f(n) = +1$. Prove that f has infinitely many zeros.

5. Let the function $f : (0, 1) \to \mathbb{R}$ be differentiable. Prove that if its derivative $f' : (0, 1) \to \mathbb{R}$ is bounded, then f is the restriction to $(0, 1)$ of a continuous function $F : [0, 1] \to \mathbb{R}$.
