First-year Analysis Examination
Part One
May 2022

Answer FOUR questions in detail.
State carefully any results used without proof.

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1. Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be bounded real sequences. Prove that

$$\liminf(a_n + b_n) \geqslant \liminf a_n + \liminf b_n$$

and show by example that the inequality can be strict.

2. Let the metric space M be compact; let $f : M \to M$ be distance-preserving and let $a \in M$. Prove that a is in the closure of the sequence $f(a), f(f(a)), \ldots$ that a generates by repeated application of f. Conclude that f is surjective.

3. Determine precisely all real numbers x for which the series

$$\sum_{n=0}^{\infty} \frac{x^n}{1 + x^n}$$

is convergent.

4. Prove that no real-valued function on the unit square $[0, 1] \times [0, 1]$ can be simultaneously continuous and injective.

5. Let $f : (0, \infty) \to \mathbb{R}$ be differentiable; let A and L be real numbers.
(i) Define what is meant by the statement $\lim_{x\to\infty} f(x) = A$.
(ii) Prove that if $\lim_{x\to\infty} f(x) = A$ and $\lim_{x\to\infty} f'(x) = L$ then $L = 0$.
