First-year Analysis Examination
Part One
January 2022

Answer FOUR questions in detail.
State carefully any results used without proof.

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1. Let (a_n) be a bounded real sequence. Let

$$U = \{u : (\exists N)(\forall n \geqslant N)(a_n < u)\}$$

and

$$L = \{\ell : (\exists N)(\forall n \geqslant N)(a_n > \ell)\}.$$

Prove that $\sup L$ and $\inf U$ exist and satisfy $\sup L \leqslant \inf U$.

2. Let X be the set of all bounded real-valued functions on the set S. Prove that the rule

$$d(f, g) = \sup\{|f(s) - g(s)| : s \in S\}$$

defines a metric on X. Are closed and bounded subsets of X compact when S is finite? Explain.

3. (i) Prove that if $(x_n)_{n=0}^{\infty}$ converges to x in a metric space then the set $\{x_n : n \in \mathbb{N}\} \cup \{0\}$ is compact.
(ii) Prove that the function $f : X \to Y$ between metric spaces is continuous if and only if its restriction $f|_K$ to each compact subset K of X is continuous.

4. Let $f : X \to Y$ be a map between metric spaces. Consider the statements:
(i) f maps Cauchy sequences to Cauchy sequences;
(ii) f is uniformly continuous.
Prove or disprove each of the implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i).

5. The function $f : (0, \infty) \to \mathbb{R}$ is differentiable and $\lim_{x\to\infty} f'(x) = b \in \mathbb{R}$.
(i) For $h > 0$ fixed, calculate the limit of $(f(x + h) - f(x))/x$ as $x \to \infty$.
(ii) Determine b if it is also known that $f(x) \to a \in \mathbb{R}$ as $x \to \infty$.
