First-year Analysis Examination
Part One
August 2021

Answer FOUR questions in detail.
State carefully any results used without proof.

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1. Let $f : (0, \infty) \to \mathbb{R}$ be continuous. Prove that if $\limsup_{n\to\infty} f(n) = 1$ and $\liminf_{n\to\infty} f(n) = -1$ then f has infinitely many zeros.

2. Let $f : X \to Y$ be a continuous bijection between metric spaces. For each of the following statements, give a proof or a counterexample, as appropriate:
(a) If X is compact then f^{-1} is continuous;
(b) if Y is compact then f^{-1} is continuous.

3. The function $f : \mathbb{R} \to \mathbb{R}$ satisfies $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to\infty} f(x) = 0$. Prove that if f is continuous then it is uniformly continuous.

4. Let $(a_n)_{n=0}^{\infty}$ be a sequence of positive reals. Prove that if the series $\sum_{n=0}^{\infty} a_n$ converges then so does the series $\sum_{n=0}^{\infty} \sqrt{a_n a_{n+1}}$. Prove also that the converse holds when $(a_n)_{n=0}^{\infty}$ is decreasing.

5. Let the function f be defined on an open interval about 0; assume that $f(0) = 0$ and that f is differentiable at 0. When n is a positive integer, determine the value of the limit

$$\lim_{x\to 0} \frac{1}{x}\big[f(x) + f(\tfrac{x}{2}) + f(\tfrac{x}{3}) + \cdots + f(\tfrac{x}{n})\big].$$
