Second Semester Algebra exam, August 2025

Answer 4 questions. You should indicate which 4 problems you wish to be graded. Write your solutions clearly in complete English sentences. You may quote theorems (within reason) as long as you state them clearly.

Name: ____

Problems: 1 2 3 4 5 6

1. Let R be a commutative ring with $1 \neq 0$.
   (a) (2 points) Define what it means for an ideal of R to be maximal.
   (b) (8 points) Prove that every proper ideal of R is contained in a maximal ideal.

2. (a) (2 points) Define the term *Euclidean domain*.
   (b) (8 points) Prove that the polynomial ring $k[x]$, where k is a field, is a Euclidean domain.

3. (a) (3 points) Give the definition of the term *unique factorization domain*.
   (b) (7 points) Show that the ring $\mathbb{Z}[\sqrt{-5}]$ is not a unique factorization domain.

4. Let k be a field, let $k[x, y]$ denote the polynomial ring in two variables over k. Let $R = k[x, y]/(xy)$ and denote the image in R of a polynomial $f \in k[x, y]$ by $\overline{f}$.
   (a) (5 points) Prove that if P is a prime ideal of R then either $\overline{x} \in P$ or $\overline{y} \in P$ (or both).
   (b) (5 points) Describe all of the prime ideals of R that contain $\overline{y}$.

5. (10 points) Using the rational canonical form, find one representative for each conjugacy class in $\mathrm{GL}(5, \mathbb{F}_2)$ whose elements have order 4.

6. (10 points) Let L be an algebraic field extension of the field K and let R be a subring of L that contains K. Prove that R is a field.