Second Semester Algebra exam, May 2025

Answer 4 questions. You should indicate which 4 problems you wish to be graded. Write your solutions clearly in complete English sentences. You may quote theorems (within reason) as long as you state them clearly.

Name: ______________________________

Problems: 1 2 3 4 5 6

1. (a) (3 points) Define the terms *Euclidean domain* and *principal ideal*.
   (b) (7 points) Prove that in a Euclidean domain every ideal is a principal ideal.
2. (10 points) Let p be a prime and $R = \{r \in \mathbb{Q} | r = \frac{a}{b} \text{ for some } a, b \in \mathbb{Z} \text{ with } p \nmid b.\}$. Prove that pR is the unique maximal ideal of R.
3. (10 points) Using the rational canonical form, find one representative for each conjugacy class in $\mathrm{GL}(4, \mathbb{F}_2)$ whose elements have order 3 or 6. (Hint: in $\mathbb{F}_2[x]$ we have $x^6 - 1 = (x^3 - 1)^2$.)
4. Determine which of the following are irreducible in the given rings.
   (a) (3 points) $X^3 - X^2 + X + 2$ in $\mathbb{Q}[X]$
   (b) (3 points) $X^4 + X + 1$ in $\mathbb{F}_2[X]$
   (c) (4 points) $Y^3 + 2X^2Y + X(X + 1)$ in $\mathbb{Q}[X, Y]$
5. Let F be a field, V a finite-dimensional F-vector space and $T \in \mathrm{End}_F(V)$ a linear operator.
   (a) (4 points) Explain why there is a F-algebra homomorphism $\psi_T : F[x] \to End_F(V)$ mapping x to T.
   (b) (2 points) Define the minimum polynomial $m_T(x)$ of T.
   (c) (4 points) Let $F[T]$ be the F-subalgebra of $\mathrm{End}_F(V)$ generated by T. Prove that if $m_T(x)$ is irreducible in $F[x]$ then every nonzero element of $F[T]$ is invertible (under composition of operators).
6. (10 points) Let $n \geq 1$ and let $F = \mathbb{Q}(\sqrt{1}, \sqrt{2}, \ldots, \sqrt{n})$. Show that $\sqrt[3]{2} \notin F$.