Answer **four** problems. (If you turn in more, the first four will be graded.) Put your answers in numerical order and circle the numbers of the four problems below your name. Within reason, you may quote theorems as long as you state them clearly.

Name: ______________________________

Problems to be graded: 1 2 3 4 5 6

1. (10 points) Show that if the ring R is a Euclidean domain, then R is a principal ideal domain.

2. (10 points) Suppose that F is a field and G is a finite multiplicative subgroup of $F\backslash\{0\}$. Prove that G is cyclic.

3. (10 points) Suppose that F is a field whose characteristic is not 2. Assume that $d_1, d_2 \in F$ are not squares in F. Prove that $F\left(\sqrt{d_1}, \sqrt{d_2}\right)$ is of dimension 4 over F if $d_1 d_2$ is not a square in F and of dimension 2 otherwise.

4. (10 points) Let $GL_4(\mathbf{Z}/3\mathbf{Z})$ denote the group of all invertible 4 by 4 matrices with entries in $\mathbf{Z}/3\mathbf{Z}$, the field of three elements. Use rational canonical forms to determine the number of conjugacy classes of elements of order 4. Give the rational canonical form for each class.

5. Let R be an integral domain, and M be an R-module.
   (a) (3 points) Define the rank of M. (Note that this definition is different from the definition of free module of some rank. In particular, it applies to modules that are not torsion free.)
   (b) (7 points) Prove that if M is a free R-module of rank n, (where n is a non-negative integer), then the rank of any submodule of M is at most n.

6. Let F be a field.
   (a) (4 points) What does it mean to say that a field extension of F is algebraic?
   (b) (6 points) Let $F \subseteq K \subseteq L$ be three fields. Prove that if K is algebraic over F and L is algebraic over K, then L is algebraic over F.