Answer **four** problems. (If you turn in more, the first four will be graded.) Put your answers in numerical order and circle the numbers of the four problems below your name. Within reason, you may quote theorems as long as you state them clearly.

Name: ____________________

Problems to be graded: 1 2 3 4 5 6

1. (10 points) Prove the following version of the Chinese remainder theorem: Let R be a commutative ring with 1 and let I, J be ideals of R such that $I + J = R$. Then $IJ = I \cap J$ and there is a ring isomorphism $R/IJ \to (R/I) \times (R/J)$.

2. Let R be a commutative ring and let M be a maximal ideal of R.
   (a) (7 points) Prove that if R has an identity then R/M is a field.
   (b) (3 points) Given an example of a commutative ring R and a maximal ideal M of R such that R/M is not a field.

3. Let R be a commutative ring with identity 1 different from 0.
   (a) (2 points) Define what is a prime ideal of R.
   (b) (1 point) Define what is a minimal prime ideal of R.
   (c) (7 points) Prove that R has a minimal prime ideal.

4. (10 points) Prove that the ring $\mathbf{Z}\left[\sqrt{-2}\right] = \left\{a + b\sqrt{-2} : a, b \in \mathbf{Z}\right\}$ is a Euclidean domain with respect to the norm defined by $N\left(a + b\sqrt{-2}\right) = a^2 + 2b^2$ for all $a, b \in \mathbf{Z}$.

5. (10 points) Determine the conjugacy classes of elements of order 8 in $GL(5, \mathbf{Q})$, and, for each conjugacy class, give a representative matrix. (You may use the fact that $x^4 + 1$ is irreducible in $\mathbf{Q}[x]$.)

6. (10 points) Let $F = \mathbf{Q}\left(\sqrt[4]{2}, \sqrt{-1}\right)$. Calculate the degree of the extension $F/\mathbf{Q}$, and justify each of the steps of your calculation.