Second Semester Algebra Exam **January 2024**

Answer four problems, and on the list below circle the problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

Grade: 1 2 3 4 5 6 7

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1. Suppose k is a field. (i) If $A = k[X,Y]/(X^2Y^3)$ find all minimal prime ideals of R. (ii) Show that $B = k[X,Y]/(X^2,Y^3)$ has a unique prime ideal, and that this prime ideal is maximal. What is it?

2. Which of the following polynomials are irreducible? Explain.

   (a) $X^3 + X + 1$ in $\mathbb{F}_2[X]$;

   (b) $X^3 + X + 1$ in $\mathbb{Q}[X]$;

   (c) $X^5 + 10X^2 + 25X + 5$ in $\mathbb{Q}[X]$;

3. Suppose R is a ring with identity and M is a nonzero left R-module. Let $\mathrm{End}_R(M)$ be the ring of R-linear maps $M \to M$. Show that if 0 and M are the only R-submodules of M, then $\mathrm{End}_R(M)$ is a division ring.

4. Let R be an integral domain. (i) Define what it means for an element of R to be *irreducible*, and to be *prime*. (ii) Show that a prime element is irreducible.

5. Suppose F is a field, V and W are F-vector spaces of finite dimension and $f : V \to W$ is a surjective linear map, and let $U = \mathrm{Ker}(f)$. Show using the definitions that $\dim_F(V) = \dim_F(W) + \dim_F(U)$.

6. Suppose L/K is an extension of fields. Show that $\alpha \in L$ is algebraic over K if and only if there is a nonzero sub-K-vector space $V \subseteq L$ of finite dimension such that $\alpha V \subseteq V$ (for "if," pick a nonzero $v \in V$ and consider $\alpha^n v$ for $n \in \mathbb{N}$).

7. Find representatives of each conjugacy class of $GL_4(\mathbb{F}_2)$ of order 3 or 5 (the order of a conjugacy class is the order of any element of it).