Answer **four** problems. (If you turn in more, the first four will be graded.) Put your answers in numerical order and circle the numbers of the four problems below your name. Within reason, you may quote theorems as long as you state them clearly.

Name: ______________________________

Problems to be graded: 1 2 3 4 5 6

1. (10 points) Show that the ring $\mathbf{Z}[\sqrt{-5}]$ is not a principal ideal domain, and give an example of a non-principal ideal in this ring.

2. (10 points) Let $R = \mathbf{Z}[\sqrt{-1}]$ be the ring of Gaussian integers. Let $M$ be a maximal ideal of $R$. Prove that $R/M$ is a finite field of order $p$ or $p^2$ where $p$ is a (rational) prime number.

3. (10 points) Let $V$ be a vector space over a field $F$ and let $S$ be a subset of $V$ such that $Span(S) = V$. Prove that there is a subset $B$ of $S$ which is a basis for $V$. (You may not assume that $S$ is finite).

4. Which of the following polynomials are irreducible? Explain.
   (a) (3 points) $X^4 + X^2 + 1$ in $\mathbf{F}_2[X]$;
   (b) (3 points) $X^3 + X^2 + 1$ in $\mathbf{Q}[X]$;
   (c) (4 points) $X^5 + 6X^2 + 4X + 18$ in $\mathbf{Q}[X]$.

5. (10 points) Find representatives of all conjugacy classes of elements of order 5 in $GL_3(\mathbf{F}_{19})$. Hint: $x^5 - 1 = (x-1)(x^2 - 4x + 1)(x^2 + 5x + 1)$ in $\mathbf{F}_{19}[x]$, and the quadratic factors are irreducible.

6. Let $E/F$ be a field extension and let $R$ be a subring of $E$ which contains $F$.
   (a) (5 points) Prove that if $E/F$ is an algebraic extension then $R$ is a field.
   (b) (5 points) Give an example where $E/F$ is not an algebraic extension and $R$ is not a field.