Second Semester Algebra Exam **August 2023**

Answer four problems, and on the list below circle the problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

Grade: 1 2 3 4 5 6

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1. Suppose R is a nonzero commutative ring with identity, and recall that $x \in R$ is *nilpotent* if $x^n = 0$ for some n (possibly depending on x). (i) Show that the set I of nilpotent elements of R is a proper ideal. (ii) Show that R/I has no nonzero nilpotent elements.

2. Which of the following polynomials are irreducible in the given ring? Explain, and factor the ones that are not irreducible.

   (a) $X^4 + 6X^3 - 9X - 21$ in $\mathbb{Q}[X]$.

   (b) $X^3 + 2X^2 + X + 1$ in $\mathbb{Z}[X]$.

   (c) $X^3 + Y^2 - Y + 1$ in $\mathbb{Z}[X, Y]$

3. Let R be an integral domain. (i) Show that any prime element of R is irreducible. (ii) Show that if R is a PID then any irreducible element is prime. (iii) Show that $\mathbb{Z}[\sqrt{-5}]$ is an integral domain, 2 is irreducible, and 2 is not prime.

4. Assume that if F is a field. Find the minimal prime ideals of the ring

$$R = F[x_1, x_2, x_3, x_4, x_5, x_6]/(x_1x_2, x_3x_4, x_5x_6).$$

   (a minimal prime ideal is a prime ideal that does not properly contain any other prime ideal).

5. Suppose L/K and E/L are algebraic extensions of fields. Show that E/K is algebraic.

6. (a) (5 points) Find representatives in rational canonical form for all conjugacy classes in $GL_6(\mathbb{Q})$ with characteristic polynomial $(X^3 - 1)^2$. (b) For each representative, give the Jordan normal form in $GL_6(\mathbb{C})$.