First Year Exam (Algebra, Semester 2), January 2023

Answer 4 questions. You should indicate which 4 problems you wish to be graded. Write your solutions clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. Prove that the following polynomials are irreducible in $\mathbb{Q}[x]$, stating clearly any theorems you wish to apply.
   (a) (3 points) $x^3 - x^2 + x + 5$.
   (b) (3 points) $x^6 - 300$.
   (c) (4 points) $x^4 - 4x^3 + 2x^2 + 21x - 7$.

2. (10 points) Show that the ring $\mathbb{Z}[\sqrt{-2}]$ is a Euclidean domain with respect to the norm $N(a + b\sqrt{-2}) = a^2 + 2b^2$. (You may regard this ring as a subring of the field of complex numbers, with $\sqrt{-2} = i\sqrt{2}$.)

3. Let R and S be commutative rings with 1 and $\phi : R \to S$ a ring homomorphism with $\phi(1) = 1$. Let I be an ideal of S and $\phi^{-1}(I) = \{r \in R \mid \phi(r) \in I\}$.
   (a) (2 points) Show that $\phi^{-1}(I)$ is an ideal of R.
   (b) (4 points) Show that if I is a prime ideal then so is $\phi^{-1}(I)$.
   (c) (4 points) If I is maximal, is $\phi^{-1}(I)$ necessarily maximal? Give a proof or counterexample.

4. Let R be a ring with 1 and M a (unital) left R-module.
   (a) (2 points) State what it means for M to satisfy the Ascending Chain Condition (i.e. for M to be Noetherian).
   (b) (3 points) Prove that if M satisfies the ACC then M is finitely generated.
   (c) (5 points) Give an example of a finitely generated R-module M that does not satisfy the ACC.

5. (10 points) Use the rational canonical form to find one representative of each conjugacy class of elements of order 6 in $\mathrm{GL}(4, \mathbb{Q})$.

6. Let $K \subset F \subset E$ be field.
   (a) (6 points) If $|F : K| = n$ and $|E : F| = m$, where m and n are finite, show that $|E : K| = nm$.
   (b) (4 points) Prove that if $|E : K|$ is finite, then $|F : K|$ and $|E : F|$ are both finite.