First Year Exam (Algebra, Semester 2), August 2022

Answer 4 questions. You should indicate which 4 problems you wish to be graded. Write your solutions clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. (a) (3 points) Give the definitions of the terms *euclidean domain* and *principal ideal domain.*

   (b) (7 points) Prove that a euclidean domain must be a principal ideal domain.

2. (10 points) Prove that a finite integral domain must be a field.

3. (10 points) Let R be a nonzero ring with 1 and I a proper ideal of R. Prove that R has a maximal ideal that contains I.

4. Let $n \geq 2$ and let R be the ring of $n \times n$ matrices with entries in a field F.

   (a) (7 points) Prove that the only 2-sided ideals of R are the zero ideal and the whole ring.

   (b) (3 points) Give an example of a nonzero, proper left ideal of R.

5. (10 points) Determine the conjugacy classes of elements of order 6 in $\mathrm{GL}(4, \mathbb{Q})$, the group of invertible 4×4 matrices with rational entries.

6. (10 points) Let p be a prime and let $\theta = e^{\frac{2\pi i}{p}} \in \mathbb{C}$, a primitive p-th root of unity. Prove that $|\mathbb{Q}(\theta) : \mathbb{Q}| = p-1$. (You may quote general criteria for irreducibility of polynomials, but irreducibility of particular polynomials must be explained.)