Second Semester Algebra Exam **May ??, 2021**

Answer four problems. You should indicate which problems you wish to have graded. Write your solutions clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. Prove the Chinese remainder theorem: Let R be a commutative ring with 1 and let I, J be ideals in R such that $I + J = R$. Then $IJ = I \cap J$ and there is a ring isomorphism $R/IJ \cong (R/I) \times (R/J)$.

2. Prove that the ring $\mathbb{Z}[\sqrt{-2}] = \{a + b\sqrt{-2} : a, b \in \mathbb{Z}\}$ is a Euclidean domain with respect to the field norm

$$\mathrm{N} : \mathbb{Z}[\sqrt{-2}] \to \mathbb{Z}_{\geq 0}, \quad \mathrm{N}(a + b\sqrt{-2}) = a^2 + 2b^2.$$

3. Prove that each of the following is irreducible in the indicated ring.
   (a) $2X^3 + 2X^2 + 3X + 1 \in \mathbb{Q}[X]$
   (b) $Y^3 + X^2 + XY - Y \in \mathbb{Q}[X, Y]$
   (c) $X^4 - X^3 + 4X^2 - 15 \in \mathbb{Z}[X]$

4. Let F be a field and let V be a vector space over F. Let $S \subset V$ be a set which spans V. Use Zorn's lemma to prove that V has a basis which is contained in S.

5. Determine the number of similarity classes of matrices in $A \in \mathrm{M}_5(\mathbb{C})$ satisfying $A^3 = 0$. Give a representative for each similarity class in Jordan canonical form.

6. (a) Let L/K be a field extension of degree 5 and let $f(X) \in K[X]$ have degree 4. Prove that if $\alpha \in L$ is a root of $f(X)$ then $\alpha \in K$.
   (b) Give an example of a field extension L/K of degree 4, a polynomial $f(X) \in K[X]$ of degree 5, and a root α of $f(X)$ such that $\alpha \in L$ but $\alpha \notin K$.