Second Semester Algebra Exam – May 2020
Department of Mathematics, University of Florida

NAME:

Problems: 1 2 3 4 5 6

Instructions: Do four out of the six problems below. Indicate on the line above the problems you have done by circling the number. If you hand in more than four problems, only the first four will be graded. You may (within reason) use results from the text if you state them fully.

1. Suppose R is a ring with identity, and recall that a nonzero unital (left) R-module M is *simple* if its only submodules are 0 and M. Suppose that M is a simple R-module.

   (i) (3 points) Show that M is cyclic, i.e. generated by a single element.

   (ii) (4 points) Show that the ring $\mathrm{End}(M)$ of all R-linear endomorphisms of M is a division ring.

   (iii) (3 points) Show that if R is commutative then $M \simeq R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m} \subset R$, and identify $\mathrm{End}(M)$ in this case.

2. Which of the following polynomials are irreducible in the given ring? Explain briefly, and if the polynomial is not irreducible, give an irreducible factorization.

   (i) (3 points) $X^3 + 5X^2 + 8X + 6 \in \mathbb{Q}[X]$

   (ii) (3 points) $X^5 + (2 + 3i)X + 13 \in \mathbb{Q}(i)[X]$

   (iii) (4 points) $X^3 + Y^3 - 1 \in \mathbb{Q}[X, Y]$

3. Let R be the ring

$$R = \{a + b\sqrt{-13} \in \mathbb{C} \mid a,\ b \in \mathbb{Z}\}.$$

   (i) (2 points) Find the units of R (introduce an appropriate norm).

   (ii) (2 points) Show that $14 \in R$ has two distinct factorizations into irreducible elements.

   (iii) (3 points) Show that $(7, 1 + \sqrt{-13})$ and $(7, 1 - \sqrt{-13})$ are prime ideals in R.

   (iv) (3 points) Show that (7) is not a prime ideal in R, and write it as a product of prime ideals.

4. Let K be a field. Suppose V is a K-vector space of finite dimension and that W_1, $W_2 \subseteq V$ are subspaces. Show that

$$\dim_K(W_1 + W_2) = \dim_K(W_1) + \dim_K(W_2) - \dim_K(W_1 \cap W_2).$$

5. Find representatives in rational canonical form for all conjugacy classes of order 5 or 7 in $GL_4(\mathbb{F}_2)$ (first find irreducible factorizations of X^5-1 and X^7-1 in $\mathbb{F}_2[X]$).

6. Suppose p and q are distinct primes in $\mathbb{Z}$ and let $\alpha = \sqrt{p}+\sqrt{q}$. Show that $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{p}, \sqrt{q})$ and use this to compute the degree of the extension $\mathbb{Q}(\alpha)/\mathbb{Q}$. Find also the minimal polynomial of α over $\mathbb{Q}$.