First Semester Algebra exam, August 2025

Answer 4 questions. You should indicate which 4 problems you wish to be graded. Write your solutions clearly in complete English sentences. You may quote theorems (within reason) as long as you state them clearly.

Name: ____

Problems: 1 2 3 4 5 6

1. (10 points) Prove that there exist at least three nonisomorphic nonabelian groups of order 12.

2. (10 points) Prove that for $n \geq 3$ the alternating group A_n is generated by 3-cycles. (You may assume that S_n is generated by transpositions.)

3. Let p be a prime, and let $G = \mathrm{GL}(3, \mathbb{F}_p)$, where $\mathbb{F}_p$ denotes the field of p elements.
   (a) (2 points) Compute the order of G.
   (b) (4 points) Let
   $$U = \{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{F}_p \}.$$
   Prove that U is a Sylow p-subgroup of G.
   (c) (4 points) Find the commutator subgroup of U.

4. Let G be a group acting on a set X.
   (a) (4 points) For $x \in X$, define the *orbit* of x and the *stabilizer* of x.
   (b) (6 points) Assume that G is finite. Prove that the size of the orbit of x divides the order of G.

5. Let G be a group.
   (a) (3 points) Define the terms *automorphism* of G, *inner automorphism* of G and *characteristic subgroup* of G.
   (b) (4 points) Prove that the set of automorphisms of G is a group in which the subset of inner automorphisms forms a normal subgroup.
   (c) (3 points) Prove that if N is a normal subgroup of G and H is a characteristic subgroup of N, then H is normal in G.

6. Let G be a group of order $675 = 3^3 5^2$.
   (a) (3 points) Prove that G has a normal Sylow 5-subgroup.
   (b) (7 points) Prove that G has an element of order 15.