Answer **four** problems. (If you turn in more, the first four will be graded.) Put your answers in numerical order and circle the numbers of the four problems below your name. Within reason, you may quote theorems as long as you state them clearly.

Name: ______________________________

Problems to be graded: 1 2 3 4 5 6

1. (10 points) State (3 points) and prove (7 points) Sylow's First Theorem.

2. Let G and H be finite groups of orders m, and n respectively.

   (a) (5 points) Prove that if m and n are coprime then

   $$Aut(G \times H) \simeq Aut(G) \times Aut(H).$$

   (b) (5 points) Give an example where m and n are not relatively prime and

   $$Aut(G \times H) \not\simeq Aut(G) \times Aut(H).$$

   Justify your claims.

3. (10 points) Here $\mathbf{Z}$ is considered as a group. Show that there are two different group homomorphisms $\phi : \mathbf{Z}/2\mathbf{Z} \to Aut(\mathbf{Z}/7\mathbf{Z})$, and prove that the corresponding semidirect products $\mathbf{Z}/7\mathbf{Z} \rtimes_\phi \mathbf{Z}/2\mathbf{Z}$ are pairwise nonisomorphic.

4. Note that $2024 = 2^3 * 11 * 23$.

   (a) (5 points) Prove that the symmetric group S_{42} has at least one element of order 2024.

   (b) (5 points) Does the alternating group A_{42} have an element of order 2024? Justify your answer.

5. Let G be a finite group.

   (a) (5 points) Define what it means to say that G is *nilpotent*.

   (b) (5 points) Without assuming any result about nilpotent groups (i.e. just from your definition), prove that if the order of G is p^n, where p is a prime and $n \in \mathbf{Z}$ with $n \geq 0$, then G is nilpotent.

6. (10 points) Let S be a finite simple group, and let H be a proper subgroup of S. Let $n = [S : H]$. Prove that S is isomorphic to a subgroup of the symmetric group S_n.