

First Semester Algebra Exam**August 2023**

Answer four problems, and on the list below circle the problems you wish to have graded. Write your answers clearly in complete English sentences.

Grade: 1 2 3 4 5 6

1. Suppose G is a group and H, K are normal subgroups of G such that $H \cap K = 1$. (i) Show that there is an injective homomorphism $G \rightarrow G/H \times G/K$. (ii) Show that if $G = HK$ then G is isomorphic to $G/H \times G/K$.
2. Suppose G is a finite group, p is a prime, $|G|$ is a power of p and $G \times S \rightarrow S$ is an action of G on S . Show that if p does not divide $|S|$ then S has an element fixed by every element of G .
3. Suppose p and q are primes such that $p < q$ and $p|q - 1$. Show that there is a unique nonabelian group of order pq , and show it is a semidirect product of two cyclic groups.
4. Give the definitions of (i) *solvable* group and (ii) *nilpotent* group. (iii) Show that A_4 is solvable but not nilpotent.
5. Show that if $n \geq 5$ then A_n has no proper subgroup of index less than n (you may assume results about the simplicity of A_n).
6. (1) (5 points) Find all conjugacy classes of elements of order 4 in S_6 . (2) (5 points) Do the same for A_6 . In both cases, justify your answer.