Department of Mathematics, University of Florida
First Semester Algebra Exam – May, 2023

Answer four problems. If you turn in more than four, only the first four will be graded. Within reason, you may use theorems as long as you state them clearly. When you are done, put the answers in numerical order, put your name in the space below and circle the numbers of the problems you wish graded.

Name:

Problems: 1 2 3 4 5 6

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1. Let $P \subset S_7$ be a Sylow 7-subgroup. Show that the normalizer N of P is a semidirect product $\mathbb{Z}/7\mathbb{Z} \rtimes \mathbb{Z}/6\mathbb{Z}$, where the homomorphism $\mathbb{Z}/6\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/7\mathbb{Z})$ is injective.

2. It is known that A_7 has a simple subgroup of order 168. Show that this subgroup is maximal.

3. Suppose G is a group of order 315. (i) Show that if G has a normal Sylow 3-subgroup, that subgroup is contained in the center of G. (ii) Deduce from this that G is abelian.

4. Suppose $\sigma \in A_8$ has order 15. (i) (7 points) Find the order of the centralizer of σ. (ii) (3 points) Find the number of conjugates of σ.

5. Suppose A is an abelian group of order p^4 where p is a prime. (i) Show that A has at most $1 + p + p^2 + p^3$ subgroups of order p. (ii) Show that equality happens if and only if every nonidentity element of A has order p.

6. Suppose G is a group and M, N are normal subgroups of G such that $G = MN$. Show that $G/(M \cap N) \simeq G/M \times G/N$.