First Semester Algebra Exam **January 2023**

Answer four problems, and on the list below circle the problems you wish to have graded. State clearly all the results from the text that you need, and write your answers clearly in complete English sentences. In all cases, be sure to justify your answer.

Grade: 1 2 3 4 5 6

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1. Denote by Q_8 the quaternion group. Show that if $f : Q_8 \to S_n$ is an injective homomorphism then $n \geq 8$.

2. Show that any group of order $80 = 2^4 \cdot 5$ is solvable (do *not* simply quote Burnside's theorem).

3. (i) (3 points) Define what it means for a group G to be *nilpotent*. (ii) (4 points) Show that any nilpotent group is solvable. (iii) (3 points) Give an example of a finite solvable group that is not nilpotent, and prove that this example is correct.

4. Find (i) (5 points) representatives of the isomorphism classes of abelian groups of order $504 = 2^3 \cdot 3^2 \cdot 7$, and (ii) (5 points) the number of elements of order 6 in each such group.

5. (i) (5 points) Find all conjugacy classes of elements of order 2 in S_6, and the number of elements in each class. (ii) (5 points) Do the same for A_6.

6. Suppose p is prime, $P \subseteq S_p$ is a Sylow p-subgroup of S_p and $N = N(P)$ is the normalizer of P. Show that

$$N \simeq (\mathbb{Z}/p\mathbb{Z}) \rtimes_{\varphi} (\mathbb{Z}/p\mathbb{Z})^{\times}$$

and describe φ.