First Year Exam (Algebra, Semester 1), August 2022

> Answer 4 questions. You should indicate which 4 problems you wish to be graded. Write your solutions clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. (10 points) Prove that the alternating group A_n, $n \geq 3$, is generated by 3-cycles. (You may assume that S_n is generated by transpositions.)

2. (a) (5 points) Prove that a group G of order 75 has a unique subgroup H of order 25.
   (b) (5 points) Show that if the above subgroup H is cyclic then G is abelian.

3. Let G be symmetric group S_n acting on $X = \{1, 2, \ldots, n\}$, $n \geq 2$.
   (a) (4 points) Explain how we have an induced action on the set $X \times X$ of ordered pairs (a, b), with a, $b \in \{1, 2, \ldots, n\}$.
   (b) (3 points) Show that there are two orbits of the action on $X \times X$ and describe them.
   (c) (3 points) Describe the stabilizer in G of an element in each of the two orbits.

4. This problem is about $G = \mathrm{GL}(3, 2)$, the group of invertible 3×3 matrices with entries from the field $\mathbb{F}_2$ of two elements.
   (a) (3 points) Compute the order of G.
   (b) (4 points) Prove that the set
   $$S = \{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{F}_2 \}$$
   forms a Sylow 2-subgroup of G.
   (c) (3 points) Find a subgroup H with $S \lneq H \lneq G$.

5. Let G be a finite group and let p be a prime. A p-subgroup of G is a subgroup whose order is a power of p.
   (a) (3 points) Prove that if P and Q are normal p-subgroups of G, then so is PQ.
   (b) (2 points) Deduce that G has a subgroup K that is a normal p-subgroup and contains every normal p-subgroup.
   (c) (3 points) Prove that G/K has no nontrivial normal p-subgroups.
   (d) (2 points) Give an example in which $|G/K|$ is divisible by p.

6. (10 points) Let G be a group and α an automorphism of G. Suppose g and h are elements of G with $\alpha(g) = h$. Prove that there exists a group $\tilde{G}$ such that G is isomorphic to a normal subgroup of $\tilde{G}$, and the images of g and h in $\tilde{G}$ are conjugate in $\tilde{G}$.