First Year Exam (Algebra, Semester 1), May 2022

Answer 4 questions. You should indicate which 4 problems you wish to be graded. Write your solutions clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. This question is about the additive group of rational numbers, denoted by $\mathbb{Q}$.
   (a) (5 points) Prove that the group $\mathbb{Q}$ has no proper subgroups of finite index.
   (b) (5 points) Prove that $\mathbb{Q}$ does not have a finite generating set.
2. (10 points) Let G be a finite group and N a normal subgroup. Show that if p is a prime and P is a Sylow p-subgroup of G, then $P \cap N$ is a Sylow p-subgroup of N.
3. Let G be a finite group.
   (a) (2 points) Define what is meant by a maximal subgroup of G.
   (b) (4 points) Let H denote the intersection of all the maximal subgroups of G. Prove that H is a characteristic subgroup of G.
   (c) (4 points) Let H be as in (b). Prove that if X is a subset of G such that $\langle X \cup H \rangle = G$ then $\langle X \rangle = G$. (Here $\langle S \rangle$ denotes the subgroup generated by the set S.)
4. (10 points) Let G be a nonabelian simple group. In $G \times G$, let $G_1 = \{(g, 1) \mid g \in G\}$ $G_2 = \{(1, g) \mid g \in G\}$. Prove that G_1 and G_2 are the only proper nontrivial normal subgroups of $G \times G$.
5. (10 points) Let S_n and A_n, denote the symmetric group and alternating group of degreee n respectively, where $n \geq 2$. Suppose σ and σ' are elements of A_n that are in the same S_n-conjugacy class. Show that they lie in the same A_n-conjugacy class if and only if there is an odd permutation in the centralizer of σ in S_n.
6. (a) (3 points) Compute the order of $\mathrm{GL}(2, 17)$, factored into prime powers.
   (b) (7 points) Prove that a group of order 7.17^2 must be abelian.