First Semester Algebra exam

Answer 4 questions. You should indicate which 4 problems you wish to be graded. Write your solutions clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. Let G be a group of order $2022 = 2 \cdot 3 \cdot 337$ (as a product of primes).
   (a) (5 points) Prove that G has a normal subgroup of index 2.
   (b) (5 points) Is there a group of order 2022 with trivial center?
2. Let G be a finite group of order p^a, where p is a prime and a is a positive integer. Prove that if N is a nontrivial normal subgroup of G, then $N \cap Z(G) \neq \{1\}$.
3. Prove that if a group G has a proper subgroup of finite index, then it has a proper normal subgroup of finite index.
4. Let G be the additive group of rational numbers.
   (a) (5 points) Prove that G is not cyclic.
   (b) (5 points) Prove that G cannot be written as the direct product of two proper subgroups.
5. Give the definition of a solvable (finite) group. (You may give any of the equivalent definitions.) Prove, using your definition, that the symmetric group on four letters is solvable.
6. Let G be a finite group and p a prime. Let K be the subgroup of G generated by all elements of order relatively prime to p.
   (a) (5 points) Prove that K is a characteristic subgroup of G.
   (b) (5 points) Prove that $|G/K|$ is a power of p.